

The Error in the Two Envelope Paradox

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The "paradox" arises in the Two Envelopes Paradox from the incorrect formulation of the argument. The information given is misused and therefore the results are incorrect for the question asked. The key is to be clear on what question we are asking. We must make sure that the question that is asked is the question that is written down.

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I. INTRODUCTION

Here is the situation: There are two identical sealed envelopes, marked "A" and "B", that each contain a different sum of money. You are handed the envelope marked A. You are told that one envelope contains twice the amount of money as the other envelope. Now you are given the choice to keep the envelope that you have or switch it for the other envelope. In some scenarios you are allowed to open the envelope that you have and count the money before making a decision. So, what do you do?

Clearly being able to count the money of one of the envelopes is irrelevant information. In other words, it does not tell you anything about the other envelope since you were not told the sum total of the two envelopes. This leaves us with the original situation described above.

The solution to the question has traditionally been formulated like this: There is a one to one symmetry between the envelopes. Thus there is a 50% chance of getting the larger amount of money. However, if we calculate the expectation of, let's say A, then one gets a very different result. Let us look at the two possibilities for A. The first is that A contains twice the amount as B ($A = 2B$). The second is that A has half of what B contains ($A = \frac{B}{2}$). Now if we calculate the expectation value we get,

$$\langle A \rangle = \frac{1}{2}(2B) + \frac{1}{2}\left(\frac{B}{2}\right) = \frac{5}{4}B \quad (1)$$

How can the expectation value of A be greater than that of B? According to this A should always be expected to hold more than B and one should *always keep* the envelope. However, if A and B are switched in the calculation the opposite occurs and one should *always switch* the envelope. Herein lies the paradox, which should you do? Keep A or trade it for B?

II. THE ERROR IN THE QUESTION

The solution to this paradox is very simple, we are asking two different questions and the second one is written

incorrectly. The first question has to do with which envelope we should keep. As far as *this* question goes, the *only* relevant information is that the two envelopes are *identical*. Whether one has more money than the other one is inconsequential because you have no way of finding out what the other envelope holds. Here is another example: Suppose someone flips a fair coin and asks you what the outcome is going to be. You have a 50/50 chance of being correct. Now you are told that the coin is unfair, or that the person who flipped the coin is able to get whichever side they want. What are your chances of being correct now? From *your* perspective, it is still 50/50! The extra information is irrelevant to you because you can only pick heads or tails. Thus, the only information that you can actually use is the fact that the coin has two sides and as such, will not influence you about which side to pick. So, from your perspective, it does not matter which one you pick.

The second question asks, "What do you expect the value of the envelope you are holding to be?" Unfortunately that is not the question that we wrote down. The error lies in what we claim are the two possibilities. The possibilities for envelope A are twice that of what envelope B *contains* or one half of what B *contains*. This is either two units or one unit, since the ratio between the two is 2:1. Here we choose the first possibility as the "two unit" possibility. The second possibility therefore must be the "one unit" possibility which is *half* of the *first* possibility. Thus the expectation should be written as follows,

$$\langle A \rangle = \frac{1}{2}(2Units) + \frac{1}{2}(Units) = \frac{3}{2}Units \quad (2)$$

Which is what one would *expect* the *average* to be, exactly one half of the total units. Expectation used in this way is, after all, an average. And, after many trials ($n \rightarrow \infty$), one would expect to get an average of 1.5 units. The point to understand here is that one cannot have both 2 units *and* a half of a unit *simultaneously*.

III. CONCLUSIONS

In the end it is not the logic that usually creates a paradox, it is the misuse or misinterpretation of it. There are several lessons to be learned from this kind of problem.

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First is that one has to be very careful that the question one is writing is indeed the question that one is asking. As has been demonstrated, this was the case. The second lesson is understanding when tools are useful and when they are not. For example look at the tossing of a die. With one toss the expectation value is 3.5 units or dots. However, this number is not even a possible outcome! But, when the die is thrown many times the *average* amount of units will indeed be 3.5. In the envelope case illustrated above, the one half that is used for the probability in the expectation value is really a *frequency*. The *probability distribution* is actually flat and therefore

one should have no expectation of what the value of the envelope is at all. On the other hand when there are many trials, as a frequency suggests, one should rightfully expect to get the average of the envelopes which is precisely what we attained.

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